HOMEWORK 3 COMPLEX ANALYSIS

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1. Problem 1

Proof. Suppose $f(z) = \sum_{n=0}^\infty a_n z^n$, $z \in \mathbb{C}$. Then, by the condition that $f(z+w) = f(z)f(w)$, we find that $f(0) = f(0)f(0) \implies f(0) = 1$.

Since we have that $f'(z) = f(z)$ we can apply induction to find that $f^{(n)}(z) = f(z)$ for all positive integers n. Now, since $f(0) = 1$, this immediately implies that $a_0 = 1$. We can differentiate our series term by term to find the following:

$$\sum_{n=0}^\infty a_n z^n = \sum_{n=0}^\infty (n+1)a_{n+1} z^n$$

Comparing the coefficients of these series yields the recursion $a_{n+1} = \frac{a_n}{n+1}$, where $a_0 = 1$. By induction we see that the sequence $a_n = \frac{1}{n!}$ satisfies this recurrence. Thus, we have:

$$f(z) = \sum_{n=0}^\infty \frac{z^n}{n!}$$

Substitute $z = x + iy$, x, $y \in \mathbb{R}$. Then, by applying the Binomial Theorem:

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$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k(iy)^{n-k}}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^k(iy)^{n-k}}{k!(n-k)!} \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{x^k(iy)^{n-k}}{k!(n-k)!} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^k(iy)^{n}}{k!n!} \\
&= \Big(\sum_{k=0}^{\infty}\frac{x^k}{k!}\Big)\Big(\sum_{n=0}^{\infty}\frac{(iy)^n}{n!}\Big) \\
&= e^x e^{iy}
\end{aligned} \tag{1.1}
$$

By Euler's formula, $e^{iy} = \cos(y) + i\sin(y)$, so we have:

$$
f(z) = e^x\big(\cos(y) + i\sin(y)\big)
$$

□

2. Problem 2

Proof. Let f be an arbitrary complex function. Then,

$$
\begin{aligned}
4\frac{\partial}{\partial \bar{z}}\frac{\partial f}{\partial z} &= 4\frac{\partial}{\partial \bar{z}}\frac{1}{2}\Big(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\Big) \\
&= \left(\frac{\partial^2 f}{\partial x^2} - \frac{1}{i}\frac{\partial^2 f}{\partial x \partial y} + \frac{1}{i}\Big(\frac{\partial^2 f}{\partial y \partial x} - \frac{1}{i}\frac{\partial^2 f}{\partial y^2}\Big)\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \Delta f
\end{aligned} \tag{2.1}
$$

Where we have used the fact that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for the last step. The case for $4\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}}$ is nearly identical. We have:

$$
\begin{aligned}
4\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}} &= 4\frac{\partial}{\partial z}\frac{1}{2}\Big(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\Big) \\
&= \Big(\frac{\partial^2 f}{\partial x^2} + \frac{1}{i}\frac{\partial^2 f}{\partial x \partial y} - \frac{1}{i}\Big(\frac{\partial^2 f}{\partial y \partial x} + \frac{1}{i}\frac{\partial^2 f}{\partial y^2}\Big)\Big) \qquad (2.2) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \Delta f
\end{aligned}
$$

□

3. Problem 3

Proof. This is immediate by the previous problem: Note that for f holomorphic, $\frac{\partial f}{\partial \bar{z}} = 0$. Thus,

$$4\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}} = 0 = \Delta f$$

Another way to see this is to note that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Since f is holomorphic we substitute the Cauchy Riemann equations, and find:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \implies \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

□

4. Problem 4

Proof. Suppose we have the $\lim \frac{|a_{n+1}|}{|a_n|} = L$. Since the sequence $|a_n|$ is just a sequence of real numbers, we use the definition of limit in the real case. Let $\epsilon > 0$. Then, there is N such that for all $n \geq N$,

$$\Big|\frac{|a_n|}{|a_{n-1}|} - L\Big| < \epsilon$$

This leads to two strict inequalities: $\frac{|a_n|}{|a_{n-1}|} < L+\epsilon$ and $\frac{|a_n|}{|a_{n-1}|} > L-\epsilon$. Since this inequality holds for any $n \geq N$, we find:

$$\frac{|a_n|}{|a_N|} = \frac{|a_n|}{|a_{n-1}|}\frac{|a_{n-1}|}{|a_{n-2}|}\dots\frac{|a_{N+1}|}{|a_N|} < (L+\epsilon)^{n-N}$$

Similarly we have that $\frac{|a_n|}{|a_N|} > (L-\epsilon)^{n-N}$. Taking nth roots leads to the following:

$$|a_n|^{\frac{1}{n}} < (L+\epsilon)^{1-\frac{N}{n}}|a_N|^{\frac{1}{n}}$$
$$|a_n|^{\frac{1}{n}} > (L-\epsilon)^{1-\frac{N}{n}}|a_N|^{\frac{1}{n}}$$

Now let $n \to \infty$, and note that taking limits does not preserve strict inequality. Since $|a_N|$ is just a fixed constant, the nth root will tend to 1. We have:

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} \leq L+\epsilon$$
$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} \geq L-\epsilon$$

Since ϵ is arbitrary, $L \leq \lim_{n\to\infty}|a_n|^{\frac{1}{n}} \leq L$, so $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = L$.

□

5. Problem 5

Proof. By taking the ratio of consecutive terms, we have the following:

$$\left|\frac{(n+1)!z^{n+1}}{n!z^n}\right| = (n+1)|z|$$

We need that this ratio is less than 1 for every positive integer n, so we have:

$$|z| < \frac{1}{n+1}$$

Letting $n \to \infty$, we find $|z| = 0$, so our radius of convergence is 0 with convergence only at the point $z = 0$.

□

6. Problem 6

Proof. Again, taking ratios of consecutive terms and setting this less than 1:

$$\left(\frac{\log(n+1)}{\log n}\right)^2 |z| < 1$$

Let $n \to \infty$. By L'Hospital's rule, we see that $\frac{\log(n+1)}{\log n} \to 1$, so we are left with $|z| < 1$, implying our radius of convergence is 1. □

7. Problem 7

Proof. Taking consecutive ratios, we want these to be less than 1 for all n. We have:

$$\frac{(1+1/n)^2(4^n+3n)}{4^{n+1}+3n+3}|z| < 1$$

Let $n \to \infty$. Then we see that $\frac{(1+1/n)^2(4^n+3n)}{4^{n+1}+3n+3} \to \frac{1}{4}$. To see this, rewrite the quantity as $\frac{1}{4}\left[\frac{(1+1/n)^2(1+3n/4^n)}{(1+3(n+1)/4^{n+1})}\right]$. Then the quantity in brackets clearly tends to 1, giving the answer.

Using this, we see that $|z| < 4$, so our radius of convergence is 4.

□

8. Problem 8

Proof. We employ Hadamard's formula:

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$$

Where R denotes the radius of convergence. Then, we have:

$$\frac{1}{R} = \limsup_{n\to\infty} \frac{|z|^2}{4(n!)^{1/n}(n+r)!^{1/n}}$$

With Stirling's formula, we know $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Substituting this asymptotic, we have:

$$\frac{1}{R} = \limsup_{n\to\infty} \frac{e|z|^2}{8\pi(n(n+r))^{1/2n}n}$$

But this limit clearly tends to 0 as $n \to \infty$, so we have that $1/R = 0$, so $R = \infty$.

Note that the formula from Problem 4 would not work because every odd term is equal to 0, making consecutive ratios undefined. □

9. Problem 9

Proof. Set $z = e^{i\phi}$, with $\phi \in \mathbb{R}$. Then, we substitute this into the given summation:

$$\sum_{n=1}^{\infty} \frac{e^{in\phi}}{n^2}$$

This sum will converge if and only if both its real and imaginary components converge. To prove this sum must converge we can use the following result from elementary analysis:

If a_n is a sequence whose series is convergent and b_n is a bounded sequence, then $\sum_{n=0}^{\infty} a_n b_n$ is also a convergent series.

Using this we can rewrite our sum as:

$$\sum_{n=1}^{\infty} \frac{\cos(n\phi)}{n^2} + i \sum_{n=1}^{\infty} \frac{\sin(n\phi)}{n^2}$$

Since both cosine and sine are bounded above by 1, and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ converges, we have that the real and imaginary parts of this sum converge, so the sum itself converges on every point of the unit circle.

□

10. Problem 10

Proof. Since we are concerned with convergence on the unit circle, we again substitute $z = e^{i\phi}$. Again we will use another elementary result from analysis which states the following: If $\sum a_n$ is a convergent series, then $a_n \to 0$. Note that the trivial cases $\phi = 0$ and $\phi = \pi$ are clearly real and divergent, so we assume $\phi \in (0, 2\pi) - \{\pi\}$. Then, by splitting into real and imaginary parts we have the following series:

$$\sum_{n=1}^{\infty} n\cos(n\phi) + i\sum_{n=1}^{\infty} n\sin(n\phi)$$

Examining nth terms, we have $n\sin(n\phi)$ and $n\cos(n\phi)$. Since $\phi \neq 0$ or π, the sequence $n\sin(n\phi)$ will never be zero and in fact oscillates arbitrarily (and likewise for $n\cos(n\phi)$ when $\phi \neq \pi/2$ or $3\pi/2$). Thus, by the result stated, its series cannot be convergent because the nth term of either sequence cannot simultaneously tend to 0, so we are done.

□